Propositional Logic

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Why Logic?

- Formalizing valid reasoning
- Used throughout mathematics, computer science
- The basis of many tools in computer science

Examples of reasoning

Which are valid?

- If it is Sunday, then I don‘t need to work.
  It is Sunday.
  Therefore I don‘t need to work.

- It will rain or snow.
  It is too warm for snow.
  Therefore it will rain.

- The Butler is guilty or the Maid is guilty.
  The Maid is guilty or the Cook is guilty.
  Therefore either the Butler is guilty or the Cook is guilty.

Elements of logic

- Which formulae are well-formed (syntax)
- What it means for formula to be true (semantics)
- When one formula follows from another one (inference)

Two possibilities: Propositional logic and First order logic

Basics of Propositional Logic

Propositions: The building blocks of propositional logic are indivisible, atomic statements (atomic propositions), for e.g.

- "The Cook is guilty"
- "It rains"
- "The girl has red hair"

and logical connectives "and", "or", "not", with which we can build **formulae**.

We are interested in knowing the following:

- When is a proposition **true**?
- When does one proposition **logically follow from** (or is logically entailed by) a knowledge base (KB), symbolically: $KB \models \varphi$?
- Can we specify a symbolic *inference relation*, notation $KB \vdash \varphi$, in such a way that it coincides with our notion of *logical entailment* ?

Syntax of Propositional Logic

Countable alphabet Σ of **atomic propositions**: *p, q, r*, …

Propositional formulae:

$\varphi, \psi \rightarrow P$		*atomic formula*
$\mid$	$\bot$	*falseness*
$\mid$	$\top$	*truth*
$\mid$	$\neg \varphi$	*negation*
$\mid$	$\varphi \wedge \psi$	*conjunction*
$\mid$	$\varphi \vee \psi$	*disjunction*
$\mid$	$\varphi \Rightarrow \psi$	*implication*
$\mid$	$\varphi \Leftrightarrow \psi$	*equivalence*

Operator precedence: $\neg > \wedge > \vee > \Rightarrow = \Leftrightarrow$. (use brackets when necessary)

Atom: atomic formula

Literal: (possibly negated) atomic formula

Clause: disjunction of literals

Intuitive Semantics

Atomic propositions can be **true** (T) or **false** (F).

The truth of a formula follows from the truth of its atomic propositions (truth assignment or interpretation) and the connectives.

Example:

$$(p \vee q) \wedge r$$

• If p and q are *false* and r is *true*, the formula is *false*

• If p and r are *true*, the formula is *true* regardless of what q is.

Semantics: Formal

A **truth assignment** of the atoms in Σ, or an interpretation over Σ, is a function I:

$$I: \Sigma \Rightarrow \{T, F\}.$$

Interpretation $I(\varphi)$ or φ^I of a formula φ:

$$I \vDash T$$

$$I \nvDash \bot$$

$$I \vDash p \text{ iff } p^I = T$$

$$I \vDash \neg \varphi \text{ iff } I \nvDash \varphi$$

$$I \vDash \varphi \wedge \psi \text{ iff } I \vDash \varphi \text{ and } I \vDash \psi$$

$$I \vDash \varphi \vee \psi \text{ iff } I \vDash \varphi \text{ or } I \vDash \psi$$

$$I \vDash \varphi \Rightarrow \psi \text{ iff if } I \vDash \varphi\text{, then } I \vDash \psi$$

$$I \vDash \varphi \Leftrightarrow \psi \text{ iff } I \vDash \varphi \text{ if and only if } I \vDash \psi$$

I **satisfies** φ (notation: $I \vDash \varphi$) or φ is **true** under I, if $I(\varphi) = T$.

Example

$$I: \begin{cases} p \rightarrow T \\ q \rightarrow F \\ r \rightarrow F \\ s \rightarrow T \\ \vdots \end{cases}$$

$$\varphi = ((p \lor q) \Leftrightarrow (r \lor s)) \land (\neg(p \land q) \lor (r \land \neg s)).$$

Question: $I \models \varphi$?

Terminology

An interpretation I is called a **model** of φ if $I \vDash \varphi$.

An interpretation is a **model** of a **set of formulae** if it fulfils all formulae of the set.

A formula φ is

- **satisfiable** if there exists I that satisfies φ,
- **unsatisfiable** if φ is not satisfiable,
- **falsifiable** if there exists I that doesn't satisfy φ, and
- **valid** (a **tautology**) if $I \vDash \varphi$ holds for all I.

Two formulae are

- **logically equivalent** ($\varphi \equiv \psi$) if $I \vDash \varphi$ iff $I \vDash \psi$ holds for all I.

Observe that all valid formulae are satisfiable and that all unsatisfiable formulae are falsifiable.

The Truth Table Method

How can we decide if a formula is satisfiable, valid, etc.?

→ Generate a truth table

Example: Is $\varphi = ((p \lor h) \land \neg h) \Rightarrow p$ valid?

p	h	$p \lor h$	$(p \lor h) \land \neg h$	$((p \lor h) \land \neg h) \Rightarrow p$
F	F	F	F	T
F	T	T	F	T
T	F	T	T	T
T	T	T	F	T

Since the formula is true for all possible combinations of truth values (satisfied under all interpretations), φ is valid.

Satisfiability, falsifiability, unsatisfiability likewise.

Some well known equivalences

Idempotence	$(\phi \wedge \phi) \equiv \phi$ $(\phi \vee \phi) \equiv \phi$
Commutativity	$(\phi \wedge \psi) \equiv (\psi \wedge \phi)$ $(\phi \vee \psi) \equiv (\psi \vee \phi)$
Associativity	$((\phi \vee \psi) \vee \gamma) \equiv (\phi \vee (\psi \vee \gamma))$ $((\phi \wedge \psi) \wedge \gamma) \equiv (\phi \wedge (\psi \wedge \gamma))$
Absorption	$(\phi \wedge (\phi \vee \psi)) \equiv \phi$ $(\phi \vee (\phi \wedge \psi)) \equiv \phi$
Distributivity	$(\phi \wedge (\psi \vee \gamma)) \equiv (\phi \wedge \psi) \vee (\phi \wedge \gamma)$ $(\phi \vee (\psi \wedge \gamma)) \equiv (\phi \vee \psi) \wedge (\phi \vee \gamma)$
Double negation	$\neg \neg \phi \equiv \phi$
De Morgan	$\neg(\phi \wedge \psi) \equiv (\neg \phi \vee \neg \psi)$ $\neg(\phi \vee \psi) \equiv (\neg \phi \wedge \neg \psi)$

Substitutability

Let ϕ and ψ be two equivalent formulae (i.e. $\phi \equiv \psi$). Let γ be a formula in which ϕ occurs as a subformula. Then γ is equivalent to γ‘ where γ‘ is obtained from γ by substituting ϕ for ψ.

Consider e.g. $p \vee \neg(q \vee r)$. Using de Morgans‘s law and substitutability, this is equivalent to $p \vee (\neg q \wedge \neg r)$.

Normal Forms

A formula is in **conjunctive normal form** (CNF) if it consists of a conjunction of disjunctions of literals $\ell_{i,j}$, i.e. if it has the following form:

$$\bigwedge_{i=1}^{n} \left(\bigvee_{j=1}^{m_i} \ell_{i,j} \right).$$

A formula is in **disjunctive normal form** (DNF) if it consists of a disjunction of conjunctions of literals:

$$\bigvee_{i=1}^{n} \left(\bigwedge_{j=1}^{m_i} \ell_{i,j} \right).$$

For every formula, there exists at least one equivalent formula in CNF and one in DNF.

A formula in DNF is satisfiable iff one disjunct is satisfiable.

A formula in CNF is valid iff every conjunct is valid.

Producing CNF

1. Eliminate $\Rightarrow$ and $\Leftrightarrow$: $\alpha \Rightarrow \beta \rightarrow (\neg\alpha \vee \beta)$ and
   $\alpha \Leftrightarrow \beta \rightarrow (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg \beta)$
2. Move $\neg$ inwards: $\neg(\alpha \wedge \beta) \rightarrow (\neg\alpha \vee \neg\beta)$ etc.
3. Distribute $\vee$ over $\wedge$: $((\alpha \wedge \beta) \vee \gamma) \rightarrow ((\alpha \vee \gamma) \wedge (\beta \vee \gamma))$
4. Simplify: $\alpha \vee \alpha \rightarrow \alpha$ etc.

The result is a conjunction of disjunctions of literals

- An analogous process converts any formula to an equivalent formula in DNF.
- During conversion, formulae can expand *exponentially*.
- Note: Conversion to CNF formula can be done *polynomially* if only satisfiability should be preserved

Logical entailment

A set of formulae (a KB) usually provides an incomplete description of the world, i.e. leaves the truth values of a proposition open.

Example: $KB = \{p \lor q, r \lor \neg p, s\}$

$$((p \lor q) \land (r \lor \neg p) \land s)$$

is definitive with respect to s, but leaves p, q, r open (though not completely!).

Models of the KB:

p	q	r	s
F	T	F	T
F	T	T	T
T	F	T	T
T	T	T	T

In all models of the KB, q $\lor$ *r* is true, i.e. q $\lor$ *r* **is logically entailed by** KB. It is a logical consequence of KB.

Logical Implication: Formal

The formula φ follows logically from the KB if φ is true in all models of the KB (symbolically KB $\models \varphi$):

$$KB \models \varphi \quad \text{iff } I \models \varphi \text{ for all models } I \text{ of KB}$$

Note: The $\models$ symbol is a *metasymbol*

Some properties of logical implication relationships:

- **Deduction theorem**: KB $\cup \{\varphi\} \models \psi$ iff KB $\models \varphi \Rightarrow \psi$
- **Contraposition theorem**: KB $\cup \{\varphi\} \models \neg\psi$ iff KB $\cup \{\psi\} \models \neg\varphi$
- **Contradiction theorem** : KB $\cup \{\varphi\}$ is unsatisfiable iff KB $\models \neg\varphi$

Question: Can we determine KB $\models \varphi$ without considering all interpretations (the truth table method)?

Proof of the Deduction Theorem

"$\Rightarrow$" Assumption: $KB \cup \{\varphi\} \models \psi$, i.e. every model of $KB \cup \{\varphi\}$ is also a model of ψ.

Let *I* be any model of KB. If *I* is also a model of φ, then it follows that *I* is also a model of ψ.

This means that *I* is also a model of $\varphi \Rightarrow \psi$, i.e. $KB \models \varphi \Rightarrow \psi$.

"$\Leftarrow$" Assumption: $KB \models \varphi \Rightarrow \psi$. Let *I* be any model of KB that is also a model of φ, i.e. $I \models KB \cup \{\varphi\}$.

From the assumption, *I* is also a model of $\varphi \Rightarrow \psi$ and thereby also of ψ, i.e. $KB \cup \{\varphi\} \models \psi$.

Proof of the Contraposition Theorem

$KB \cup \{\varphi\} \models \neg\psi$

iff $KB \models \varphi \Rightarrow \neg\psi$ (1)

iff $KB \models (\neg\varphi \lor \neg\psi)$

iff $KB \models (\neg\psi \lor \neg\varphi)$

iff $KB \models \psi \Rightarrow \neg\varphi$

iff $KB \cup \{\psi\} \models \neg\varphi$ (2)

Note: (1) and (2) are applications of the deduction theorem.

Inference Rules, Calculi and Proofs

We can often **derive** new formulae from formulae in the KB. These new formulae should **follow logically** from the syntactical structure of the KB formulae.

Example: If the KB is {…, ($\varphi \Rightarrow \psi$), …, φ, …}, then ψ is a logical consequence of KB

→Inference rules

Modus Ponens $$\frac{\varphi,\ \varphi \Rightarrow \psi}{\psi}$$

Modus Tolens $$\frac{\neg\psi,\ \varphi \Rightarrow \psi}{\neg\varphi}$$

And Elimination $$\frac{\varphi \wedge \psi}{\varphi}$$ **or** $$\frac{\varphi \wedge \psi}{\psi}$$

And Introduction $$\frac{\psi,\ \varphi}{\varphi \wedge \psi}$$

Or Introduction $$\frac{\psi}{\varphi \vee \psi}$$

Derivation: Example

A derivation of ϕ from *KB* is a sequence of sentences in which each sentence is either a member of *KB* or is the result of applying a rule of inference to elements earlier in the sequence.

Let *KB* contain the following sentences:

p

$p \Rightarrow q$

$p \Rightarrow r$

$q \wedge r \Rightarrow s$

Give a derivation of $s \wedge r$ from *KB*.

Derivation: Example (cont.)

Give a derivation of s $\wedge$ r from *KB*.

1) p (KB)
2) p $\Rightarrow$ q (KB)
3) p $\Rightarrow$ r (KB)
4) q $\wedge$ r $\Rightarrow$ s (KB)
5) q (1,2,MP)
6) r (1,3,MP)
7) q $\wedge$ r (5,6,AI)
8) s (7,4,MP)
9) s $\wedge$ r (8,6,AI)

Inference procedures specify how to compute with inference rules.

Provability

A proof of a sentence ϕ from a database *KB* is a finite sequence of sentences in which ϕ is an element of the sequence (usually the last) and every element is a member of *KB*, a logical axiom or the result of applying modus ponens to sentences earlier in the sequence.

Proofs allow only one rule of inference:. **Modus Ponens**. In addition they use logical axioms (tautologies). We will use *axiom schemata*, i.e. sentence patterns with pattern variables. Consider $\phi \Rightarrow (\psi \Rightarrow \phi)$. This makes abstraction of $p \Rightarrow (q \Rightarrow p)$ but also of $(r \lor q) \Rightarrow (t \Rightarrow (r \lor q))$.

Implication introduction	$\phi \Rightarrow (\psi \Rightarrow \phi)$
Implication distribution	$(\phi \Rightarrow (\psi \Rightarrow \gamma)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \gamma))$
Contradiction schema	$(\psi \Rightarrow \neg\phi) \Rightarrow ((\psi \Rightarrow \phi) \Rightarrow \neg\psi)$
	$(\neg\psi \Rightarrow \neg\phi) \Rightarrow ((\neg\psi \Rightarrow \phi) \Rightarrow \psi)$

Example Proof

Prove $p \Rightarrow r$ from $p \Rightarrow q$ and $q \Rightarrow r$.

1. $p \Rightarrow q$ (KB)
2. $q \Rightarrow r$ (KB)
3. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ II
4. $p \Rightarrow (q \Rightarrow r)$ (2,3,MP)
5. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ ID
6. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ (4,5,MP)
7. $p \Rightarrow r$ (1,6,MP)

Soundness and Completeness

In the case where in the calculus *C* there is a proof for a formula φ, we write

$$\mathrm{KB} \vdash_C \varphi$$

(optionally without subscript *C*).

A calculus *C* is **sound** (or **correct**) if all formulae that are derivable from a KB actually follow logically.

$$\mathrm{KB} \vdash_C \varphi \text{ implies } \mathrm{KB} \models \varphi$$

This normally follows from the soundness of the inference rules and the logical axioms.

A calculus is **complete** if every formula that follows logically from the KB is also derivable with *C* from the KB:

$$\mathrm{KB} \models \varphi \text{ implies } \mathrm{KB} \vdash_C \varphi$$

Resolution: Idea

We want a way to **derive** new formulae that does not depend on testing every interpretation.

Idea: We attempt to show that a set of formulae is unsatisfiable.

Condition: All formulae must be in CNF.

But: In most cases, the formulae are close to CNF (and there exists a fast satisfiability-preserving transformation).

Nevertheless: In the **worst case**, this derivation process requires an exponential amount of time (this is, however, probably unavoidable).

Resolution: Representation

Assumption: All formulae in the KB are in CNF.

Equivalently assumption: KB is a *set of clauses*.

Due to commutativity, associativity, and idempotence of $\vee$, **clauses** can also be understood as **sets of literals**. The **empty set of literals** is denoted by .

Set of clauses: Δ

Set of literals: C, D

Literal: ℓ

Negation of a literal: $\bar{\ell}$

An interpretation I satisfies C iff there exists $\ell \in C$ such that $I \models \ell$.
I satisfies Δ if for all $C \in \Delta$: $I \models C$. i.e. $I \not\models$, $I \not\models \{\ \}$, $I \models \{\}$, for all I.

The Resolution Rule

$$\frac{C_1 \cup \{\ell\},\ C_2 \cup \{\bar{\ell}\}}{C_1 \cup C_2}$$

$C_1 \cup C_2$ are called **resolvents** of the **parent clauses** $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$. ℓ and $\bar{\ell}$ are the **resolution literals**.

Example: {a,b,¬c} resolves with {a,d,c} to {a,b,d}.

Note: The resolvent is not equivalent to the parent clauses, but it follows from them!

Notation: $R(\Delta) = \Delta \cup \{C \mid C$ is a resolvent of two clauses from $\Delta\}$

Resolution derivations

We say D can be **derived** from Δ using resolution, i.e.

$$\Delta \vdash D,$$

If there exist $C_1, C_2, C_3, \ldots, C_n = D$ such that

$$C_i \in R(\Delta \cup \{C_1, \ldots, C_{i-1}\}), \text{ for } 1 \leq i \leq n.$$

Lemma (**soundness**) If $\Delta \vdash D$, *then* $\Delta \vDash D$.

Proof idea: Since all $D \in R(\Delta)$ *follow logically from* Δ, the lemma results through induction over the length of the derivation.

Completeness?

Is resolution also complete? i.e. is

$$\Delta \models \varphi \text{ implies } \Delta \vdash \varphi$$

valid? Only for clauses. Consider:

$$\{\{a,b\}, \{\neg b,c\}\} \models \{a,b,c\}$$

$$\nvdash \{a,b,c\}$$

But it can be shown that resolution is **refutation-complete**:

$$\Delta \text{ is unsatisfiable implies } \Delta \vdash \quad .$$

Theorem: Δ is unsatisfiable iff $\Delta \vdash$.

i.e. With the help of the contradiction theorem, we can show that KB $\models \varphi$.

A resolution trace

1. {p} (Δ)
2. {¬p,q} (Δ)
3. {¬q,r} (Δ)
4. {¬r} (Δ)

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5. {q} (1,2)
6. {¬p,r} (2,3)
7. {¬q} (3,4)

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8. {r} (3,5)
9. {r} (1,6)
10. {¬p} (4,6)
11. {¬p} (2,7)
12. {} (5,7)

Proof by Refutation

To prove that KB $\models \varphi$ prove that KB $\cup \neg \varphi$ is unsatisfiable.

- Different steps:
  1. Reformulate *KB* in CNF
  2. Reformulate $\neg \varphi$ in CNF
  3. Show that *KB* $\cup \neg \varphi$ (= Δ) is unsatifiable

Example

Assume we wish to show that $p \Rightarrow q, q \Rightarrow r \models p \Rightarrow r$

After step 1) and 2)

1. $\{\neg p, q\}$ (KB)
2. $\{\neg q, r\}$ (KB)
3. $\{\neg r\}$ $(\neg \varphi)$
4. $\{p\}$ $(\neg \varphi)$

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5. $\{\neg p, r\}$ (1,2)
6. $\{q\}$ (1,4)
7. $\{\neg q\}$ (2,3)

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8. $\{\}$ (6,7)

Illustration: Bloodtype

1. If test T is positive, then the person has blood type A or AB
2. If test S is positive, then the person has blood type B or AB
3. If a person has type A, then test T will be positive
4. If a person has type B, then test S will be positive
5. If a person has type AB, then both tests T and S will be positive
6. A person has type A, B, AB, or O (exactly one type)

Suppose T is true and S is false for a given person.

a. Show that the person must belong to bloodgroup A or O.

Illustration: Bloodtype (cont.)

Let KB be:

1. $\neg t \vee a \vee ab$ (or in implication form $t \Rightarrow a \vee ab$)
2. $\neg s \vee b \vee ab$ (or in implication form $t \Rightarrow b \vee ab$)
3. $\neg a \vee t$
4. $\neg b \vee s$
5. $\neg ab \vee s$
   $\neg ab \vee t$
6. $a \vee b \vee ab \vee o$
   $\neg a \vee \neg b$
   $\neg a \vee \neg ab$
   $\neg a \vee \neg o$
   $\neg b \vee \neg ab$
   $\neg b \vee \neg o$
   $\neg ab \vee \neg o$

a. t
   $\neg s$

Resolution: Perspective

Resolution is a complete inference procedure. Ther are other ones (Davis-Putnam, Tableax methods, ...)

For an implementation of resoltuion, we must use a strategy which determines which resolution steps are taken. Various strategies exist, we will see these later on.

In the worst case, a resoltuion proof can have exponential length. This is likely to hold for the other methods as well

For CNF formulae in propositional logic the Davis-Putnam is the fastest complete procedure that can be regarded as kind of resolution procedure.

Conclusions

- Propositional formulaes can be tautologies, satisfiable, or unsatisfiable
- Important is the concept of logical consequence or logical entailment
- The logical consequence relation can be machanised using logical inference rules. ↗Resolution